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Poisson algebras for some generalized eigenvalue problems

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Abstract
We introduce particular Poisson algebras designed to describe (after quantization) generalized eigenvalue problems for two tridiagonal matrices. These algebras are naturally implemented upon various elliptic curves.

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1. Introduction

Classical physics of conservative systems is described by the Hamilton dynamics where symplectic manifolds $M = \{q_i, p_i\}$, $i = 1, \ldots, d$, serve as phase spaces [1]. The fundamental role is played by the Poisson bracket

$$\{X, Y\} = \sum_{i=1}^{d} \left(\frac{\partial X}{\partial q_i} \frac{\partial Y}{\partial p_i} - \frac{\partial X}{\partial p_i} \frac{\partial Y}{\partial q_i} \right) \tag{1.1}$$

defined for arbitrary dynamical variables X, Y given by analytical functions on $M : X = X(\mathbf{q}, \mathbf{p})$, $Y = Y(\mathbf{q}, \mathbf{p})$. Variables $Q_i = Q_i(\mathbf{q}, \mathbf{p})$, $P_i = P_i(\mathbf{q}, \mathbf{p})$ satisfying

$$\{Q_i, P_j\} = \delta_{ij} \qquad \{Q_i, Q_j\} = \{P_i, P_j\} = 0 \tag{1.2}$$

are called canonical (e.g., the initial coordinates q_i and momenta p_i are canonical). Any transformation $(q_i, p_i) \to (Q_i, P_i)$ preserving algebraic relations (1.2) is called the canonical transformation.

Poisson brackets $\{X, Y\}$ satisfy a number of properties, which may be used for an axiomatic definition of the Hamilton dynamics. Namely, they are (1) linear in their arguments, $\{aX_1 + bX_2, Y\} = a\{X_1, Y\} + b\{X_2, Y\}$, $a, b \in \mathbb{C}$, (2) antisymmetric $\{X, Y\} = -\{Y, X\}$, (3) obey the Leibnitz rule $\{X_1X_2, Y\} = X_1\{X_2, Y\} + X_2\{X_1, Y\}$, and, finally, (4) they satisfy the Jacobi identity

$$\{X, \{Y, Z\}\} + \{Y, \{Z, X\}\} + \{Z, \{X, Y\}\} = 0.$$

We call a *Poisson algebra* any complete closed set of relations between abstract dynamical variables $X_1, \ldots, X_N$ defined through abstract Poisson brackets satisfying the listed properties.

Quantization consists in the replacement of commuting variables X, Y by operators $\hat{X}, \hat{Y}$ and of the Poisson brackets by commutators $\{X, Y\} \to [\hat{X}, \hat{Y}] \equiv \hat{X}\hat{Y} - \hat{Y}\hat{X}$. For a pair of canonical variables one sets $[\hat{q}, \hat{p}] = 1$ and the simplest (coordinate) realization of this relation is $\hat{q} = q$, $\hat{p} = -\mathrm{d}/\mathrm{d}q$. For notational convenience, we assume that $q, p \in \mathbb{C}$, which allows us to remove the Planck constant $\hbar$ and the imaginary unit $\mathrm{i} = \sqrt{-1}$ from the standard canonical commutation relation $[\hat{q}, \hat{p}] = \mathrm{i}\hbar$ by renormalization of $\hat{q}$ or $\hat{p}$.

Exponentials of the classical momentum, such as e^p, become shift operators after the quantization, $\mathrm{e}^{\hat{p}} f(q) = f(q-1)$. Suppose that a set of classical dynamical variables $X_i(q, p)$ satisfies a simple Poisson algebra. The quantization procedure is not obliged to preserve the structure of this algebra, that is replacement of q, p by operators $\hat{q}, \hat{p}$ in $X_i(q, p)$ (using some particular ordering) and of Poisson brackets by commutators will not lead, in general, to simple quantum algebras. In some cases algebraic structures are preserved without difficulties, see, e.g., [20, 7]. However, as shown in [23], the 'symmetry preserving quantization' can require a very intricate complication of the functions used for the definition of the quantum variables $\hat{X}, \hat{Y}$.

Starting from the pioneering work of Sklyanin [20] it was recognized that dynamical algebras with nonlinear Poisson bracket relations and their quantum (operator) versions play a crucial role in the description of important physical systems. The quadratic Sklyanin algebra provided a concrete example of such 'classical' and 'quantum' algebras. It appeared naturally in exactly solvable models of statistical physics and quantum field theory in the framework of the R-matrix approach developed by Faddeev and his school in Leningrad.

In [29, 7], it was shown that another nonlinear algebra with three generators, called the Askey–Wilson algebra, plays an important role in many physical systems with hidden symmetries (e.g., the Coulomb and oscillator spectral problems in spaces with constant curvature, the theory of Clebsch–Gordan and Racah coefficients, etc). The notion of Leonard duality was introduced by Terwilliger [27] for the case of finite-dimensional matrices. In [28], it was shown that the Askey–Wilson algebra characterizes such a duality for matrices. Recently, in [31] it was shown that the Poisson version of this algebra describes classical systems satisfying the Leonard duality.

The Leonard duality is closely connected with the property of bispectrality [4] for orthogonal polynomials. If we suppose that a finite-dimensional system of orthogonal polynomials $P_n(x)$ satisfies simultaneously both a three-term recurrence relation with respect to the degree of polynomials n and a difference equation with respect to the argument x on some grid x_s, then, as shown by Leonard [12], $P_n(x)$ coincide with the q-Racah polynomials discovered in [3] or their descendants. The q-Racah polynomials and their continuous measure generalizations, called the Askey–Wilson polynomials, form the most general family of 'classical' special functions among orthogonal polynomials.

In this paper we consider an analogue of the Leonard duality for a more complicated situation when the ordinary eigenvalue problem is replaced by the generalized eigenvalue problem (GEVP). GEVP problems arise naturally in physics and mathematics, e.g., after a separation of variables in partial differential equations [2]. In [30], it was shown that the GEVP for two generic tri-diagonal matrices is equivalent to the theory of biorthogonal rational functions $R_n(x)$. In the terminology of [13], Askey–Wilson polynomials are orthogonal on trigonometric (or, 'q-quadratic') grids, which were believed for some time to be the most general admissible grids (see, e.g., [15] for a discussion of biorthogonal rational functions living on this grid). In [24–26] explicit examples of biorthogonal rational functions were constructed which, being biorthogonal on *elliptic grids*, generalized previously known examples of Racah-type systems in a crucial way (their continuous measure analogues were built in [22]). These elliptic biorthogonal rational functions are closely related to the 'elliptic 6j-symbols' discovered in the theory of elliptic solutions of the Yang–Baxter equation, see the major clarifying work of Frenkel and Turaev [6] and references therein. They possess an analogue of the Leonard duality: they satisfy both a three-term recurrence relation with respect to n and a difference equation with respect to x on some elliptic grid x_s. It should be stressed that in this case we deal with GEVP rather than with ordinary eigenvalue problems. So, it is reasonable to call this property the generalized Leonard duality (GLD).

It is therefore reasonable to study the most general class of functions obeying the GLD property for two tridiagonal (Jacobi) matrices. In the quantum case (i.e., when we deal with the operators) the problem is still open, being quite complicated. We restrict ourselves only to its classical physics analogue. The main result of our paper consists in the description of algebraic relations characterizing GLD in the Poisson algebra setting. We show that all dynamical variables (the classical potentials and the grid) are expressed in terms of elliptic functions which imitate the elliptic biorthogonal rational functions. A brief announcement of this and some other results were given in [10].

The paper is organized as follows. In section 2 we recall basic results concerning classical Leonard pairs (equivalently, the classical Leonard duality) obtained earlier in [31]. In section 3 we formulate the GLD for both quantum and classical cases. In section 4 we first consider a more simple case of 'symmetric' dynamical variables. We derive a simple quadratic Poisson relation which is necessary and suffiicent for GLD in this case. In section 5 we return to the general case of the classical GLD and analyse algebraic relations for dynamical variables. In section 6 we derive corresponding potentials and show that they are expressed in terms of elliptic functions of second order in analogy with the functions of [24]. Finally, in section 7 we show that in the symmetric case the obtained algebraic relations for the classical GLD are related to some generalized Sklyanin algebras.

2. Algebraic relations for classical Leonard pairs

Throughout the paper we work only with the classical mechanical systems of one degree of freedom, $d = 1$. We suppose that $X(q, p)$ and $Y(q, p)$ are two independent dynamical variables of canonical variables q and p, $\{q, p\} = 1$. As usual, functions X, Y are called independent if in some domain of interest of the phase space (q, p) they satisfy the condition

$$\frac{\partial(X, Y)}{\partial(q, p)} \equiv \frac{\partial X}{\partial q}\frac{\partial Y}{\partial p} - \frac{\partial X}{\partial p}\frac{\partial Y}{\partial q} = \{X, Y\} \neq 0 \tag{2.1}$$

where $\partial(X, Y)/\partial(q, p)$ is the Jacobian of a change of variables.

According to the definition proposed in [11, 31], two independent variables X and Y are said to form a *classical Leonard pair* (CLP) if there exist two different canonical transformations $(q, p) \to (x, y)$ and $(q, p) \to (\xi, \eta)$ such that the first transformation brings X and Y to the form

$$X = \varphi(x) \qquad Y = A_1(x)\,\mathrm{e}^{y} + A_2(x)\,\mathrm{e}^{-y} + A_3(x) \tag{2.2}$$

and in the second case we have the representation

$$X = B_1(\xi)\,\mathrm{e}^{\eta} + B_2(\xi)\,\mathrm{e}^{-\eta} + B_3(\xi) \qquad Y = \psi(\xi) \tag{2.3}$$

where (x, y) and (ξ, η) are canonical pairs (i.e., $\{x, y\} = \{\xi, \eta\} = 1$) and $\varphi(x)$, $A_i(x)$, $\psi(\xi)$, $B_i(\xi)$ are some functions. Using canonical transformations $y \to \kappa y$, $x \to x/\kappa$ and taking the limit $\kappa \to 0$ one can obtain from (2.2) the limiting form $Y = a_1(x)y^2 + a_2(x)y + a_3(x)$.

Therefore, we shall assume that CLP admit such degenerate forms of Y in (2.2) (or of X in (2.3)) without further reservations.

It is convenient to introduce the variable

$$Z = \{X, Y\}. \tag{2.4}$$

We assume that there exists a region of values of X, Y where X and Y form indepedent variables, that is $Z \neq 0$. The latter means that in this domain we can invert the changes of variables and find $x = x(X, Y)$, $y = y(X, Y)$. As a result, we can consider Z as a function of X and Y, $Z = Z(X, Y)$. The condition that X and Y form a CLP allows us to establish the explicit form of this function $Z(X, Y)$. As shown in [11] there exist nine arbitrary constants $\alpha_{ik},\ i, k = 0, 1, 2$, such that

$$Z^2 = \sum_{i,k=0}^{2} \alpha_{ik} X^i Y^k \equiv -F(X, Y). \tag{2.5}$$

Vice versa, starting from condition (2.5) for arbitrary α_{ik} one can arrive at a CLP (including its degenerate form mentioned above). The condition $F = 0$ determines the region of the phase space with complex values of q, p where such a consideration breaks down.

From (2.5) it follows that the dynamical variables X, Y and $Z = \{X, Y\}$ form a Poisson algebra with the defining relations (2.4) and

$$\{Z, X\} = \frac{1}{2}\frac{\partial F(X, Y)}{\partial Y} \qquad \{Y, Z\} = \frac{1}{2}\frac{\partial F(X, Y)}{\partial X} \tag{2.6}$$

which are known as the classical Askey–Wilson algebra relations [7]. This algebra generates relation (2.5) with the interpretation of the constant α_{00} as a value of the corresponding Casimir element [11]. In this way we obtain a particular example of the quadratic algebras, the most popular representative of which is given by the Sklyanin algebra [20]. In a more general setting, algebraic relations between dynamical variables involve polynomials of generators (see, e.g., particular polynomial quantum algebras in [21] derived with the help of generalized supersymmetry [18]).

Suppose that X is the Hamiltonian of some physical system. Then the first canonical transformation $(q, p) \to (x, y)$ is, in fact, an action-angle transformation: it maps X into a function depending on only one canonical variable x. Similarly, canonical transformation $(q, p) \to (\xi, \eta)$ is an action-angle variables transformation for a system with the Hamiltonian Y. Existence of a CLP can be considered as some duality property of two Hamiltonians with respect to prescribed dependence on the momenta y and η of the 'conjugated' Hamiltonians (i.e., Y and X, respectively). From this point of view, the CLP property is equivalent to the notion of duality discussed in the theory of integrable systems, see, e.g., [19, 5].

We note that the quantum analogue of the CLP property coincides with the standard Leonard duality [12] which results in the characterization of the q-Racah polynomials as the most general self-dual orthogonal polynomials. As shown in [29], the quantum analogue of the algebra (2.6) describes these polynomials via the representation theory (see also [27] for similar algebraic treatments). Its relation to the standard $sl_q(2)$ quantum algebra was described in [8, 9].

3. Duality for a generalized eigenvalue problem

The main motivation of this paper consists in the following. In [24–26], we described a family of discrete biorthogonal rational functions $R_n(z)$ and $T_n(z)$, $n = 0, \ldots, N-1$, satisfying the property

$$\sum_{s=0}^{N-1} w_s R_n(z_s) T_m(z_s) = h_n \delta_{nm} \tag{3.1}$$

with some weight function w_s and normalization constants h_n. The sequence z_s is called the 'grid' and is expressed in terms of the Jacobi theta functions. Both $R_n(z_s)$ and $T_n(z_s)$ satisfy three term recurrence relations in the variable n and second order difference equations in s.

Let us introduce a set of N vectors $\Phi_s = (R_0(z_s), R_1(z_s), \ldots, R_{N-1}(z_s))^t$, $s = 0, \ldots, N-1$. Then there exist two tridiagonal matrices L_1, L_2 such that

$$L_1 \Phi_s = z_s L_2 \Phi_s \tag{3.2}$$

where it is assumed that the matrices $L_{1,2}$ act on the vector Φ_s in the standard manner. Similarly, we introduce a set of N vectors $\Psi_n = (R_n(z_0), R_n(z_1), \ldots, R_n(z_N))^t$, $n = 0, \ldots, N-1$. Then there exist two tridiagonal matrices M_1, M_2 such that

$$M_1 \Psi_n = \lambda_n M_2 \Psi_n \tag{3.3}$$

for some sequence of numbers λ_n (the dual 'grid'). Since the functions $R_n(z_s)$ satisfy simultaneously two generalized eigenvalue problems (3.2) and (3.3), it is natural to consider the following problem.

Let X and Y be two invertible $N \times N$ matrices with different eigenvalues λ_k and μ_k, $k = 0, \ldots, N-1$. We denote as ϕ_k and ψ_k linearly independent eigenvectors of X and Y, respectively

$$X\phi_k = \lambda_k \phi_k \qquad Y\psi_k = \mu_k \psi_k. \tag{3.4}$$

Now we assume that in the basis of vectors ψ_k the matrix X takes the form

$$X\psi_k = X_2^{-1} X_1 \psi_k \tag{3.5}$$

where X_1, X_2 are two tridiagonal matrices, that is

$$\begin{aligned} X_1\psi_k &= \alpha_{k+1}^{(1)} \psi_{k+1} + \beta_k^{(1)} \psi_k + \gamma_k^{(1)} \psi_{k-1} \\ X_2\psi_k &= \alpha_{k+1}^{(2)} \psi_{k+1} + \beta_k^{(2)} \psi_k + \gamma_k^{(2)} \psi_{k-1}. \end{aligned} \tag{3.6}$$

In the same way, we assume that there exist two tridiagonal matrices Y_1, Y_2 such that

$$Y\phi_k = Y_2^{-1} Y_1 \phi_k \tag{3.7}$$

with the properties

$$\begin{aligned} Y_1\phi_k &= \xi_{k+1}^{(1)} \phi_{k+1} + \eta_k^{(1)} \phi_k + \zeta_k^{(1)} \phi_{k-1} \\ Y_2\phi_k &= \xi_{k+1}^{(2)} \phi_{k+1} + \eta_k^{(2)} \phi_k + \zeta_k^{(2)} \phi_{k-1}. \end{aligned} \tag{3.8}$$

It is interesting to classify all matrices X and Y admitting such a GLD property. This problem implies an explicit description of algebraic structures behind such a construction that generalize the Askey–Wilson algebra. We conjecture that a complete solution of this problem yields discrete biorthogonal rational functions of [24–26]. Here we consider a Poisson algebra analogue of this problem.

We take two independent functions $X(q, p)$ and $Y(q, p)$ of canonical variables q, p and suppose that there exist canonical transformations $(q, p) \to (x, y)$ and $(q, p) \to (\xi, \eta)$ such that the first transformation leads to

$$X = \varphi(x) \qquad Y = \frac{Y_1(x, y)}{Y_2(x, y)} \tag{3.9}$$

and in the second case we have

$$Y = \psi(\xi) \qquad X = \frac{X_1(\xi,\eta)}{X_2(\xi,\eta)} \tag{3.10}$$

where $X_r, Y_r, r = 1, 2$, are some classical 'tridiagonal functions', that is

$$\begin{aligned} X_r(\xi,\eta) &= A_1^{(r)}(\xi)\,\mathrm{e}^{\eta} + A_2^{(r)}(\xi)\,\mathrm{e}^{-\eta} + A_0^{(r)}(\xi) \\ Y_r(x,y) &= B_1^{(r)}(x)\,\mathrm{e}^{y} + B_2^{(r)}(x)\,\mathrm{e}^{-y} + B_0^{(r)}(x). \end{aligned} \tag{3.11}$$

We call the pair (X, Y) satisfying such a property the generalized CLP.

4. Symmetric Jacobi matrices case

If we suppose that $A_0^{(r)}(x) = B_0^{(r)}(x) = 0$, which corresponds to the classical mechanical analogues of the symmetric ('two-diagonal') Jacobi matrices, then the problem is easily solvable.

Let us introduce a new variable $Z = \{X, Y\}$. In the representation (3.9), we can rewrite Y as

$$Y = e_0(x) + \frac{e_1(x)}{\mathrm{e}^{2y} - g(x)} \tag{4.1}$$

where $e_0(x), e_1(x), g(x)$ are some functions. The choice $g(x) = 0$ leads to a subcase of the standard CLP, therefore we assume that $g(x) \neq 0$. Straightforward computations yield

$$\begin{aligned} Z &= -2\varphi'(x)\frac{e_1(x)\,\mathrm{e}^{2y}}{(\mathrm{e}^{2y} - g(x))^2} \\ &= -2\varphi'(x)(Y - e_0(x))\left(1 + \frac{g(x)(Y - e_0(x))}{e_1(x)}\right). \end{aligned}$$

On the one hand, we thus obtain

$$Z = u_2(X)Y^2 + u_1(X)Y + u_0(X) \tag{4.2}$$

for some functions $u_i(X), i = 1, 2, 3$. On the other hand, analogous consideration of (3.10) yields

$$Z = v_2(Y)X^2 + v_1(Y)X + v_0(Y) \tag{4.3}$$

for some other functions $v_i(Y)$. Comparing (4.2) and (4.3), we immediately arrive at the general expression for Z in terms of X, Y

$$Z = \sum_{i,k=0}^{2} \alpha_{ik} X^i Y^k \tag{4.4}$$

for some coefficients α_{ik}.

Theorem 1. *Dynamical variables X and Y form a generalized symmetric CLP, that is they admit representations (3.9) and (3.10) with $A_0^{(r)} = B_0^{(r)} = 0, r = 1, 2,$ if and only if their Poisson bracket $Z = \{X, Y\}$ takes the form (4.4) with nine arbitrary coefficients α_{ik}.*

Proof. The 'only if' part of this statement has been proved already. It remains to show that expression (4.4) for arbitrary α_{ik} results in (3.9) and (3.10) after some canonical transformations.

Let us choose a representation $X = q, Y = F(q, p)$ with some unknown function $F(q, p)$. Then, we obtain the following simple differential equation for $F(q, p)$:

$$Z = \frac{\partial F(q,p)}{\partial p} = \sum_{i,k=0}^{2} \alpha_{ik} q^i F^k(q,p)$$

which has a general solution of the form

$$F(q,p)=\frac{A_1^{(1)}(q)\,\mathrm{e}^{\omega(q)(p-p_0(q))}+A_2^{(1)}(q)\,\mathrm{e}^{-\omega(q)(p-p_0(q))}}{A_1^{(2)}(q)\,\mathrm{e}^{\omega(q)(p-p_0(q))}+A_2^{(2)}(q)\,\mathrm{e}^{-\omega(q)(p-p_0(q))}}$$

for some known functions $A_1^{(i)}(q)$, $\omega(q)$ and an arbitrary function $p_0(q)$. We perform now a change of variables: $x=x(q)$ and $y=\omega(q)(p-p_0(q))$, where $x'(q)\omega(q)=1$. As a result, replacing in X and Y q-dependence by x-dependence via inversion of the function $x=x(q)$, we obtain the desired expressions (3.9) with $A_0^{(r)}=0$.

Evidently, the same procedure applies to derivation of the general admissible form of X and Y in the representation (3.10). □

Now we fix the coefficients α_{ik} and find explicit forms of the functions $\varphi(x)$, $e_0(x)$, $e_1(x)$ and $g(x)$. First of all, we note that the simple canonical transformation $x\to x$, $y\to y+f(x)$ with an appropriate function $f(x)$ allows us to reduce $g(x)$ to a non-zero constant, which we set equal to 1. Then, from the equation

$$\sum_{i,k=0}^{2}\alpha_{ik}X^iY^k+2\varphi'(x)e_1(x)\frac{\mathrm{e}^{2y}}{(\mathrm{e}^{2y}-1)^2}=0$$

we obtain three relations

$$2\varphi'(x)=-e_1(x)\pi_2(\varphi)\qquad 2(2e_0(x)-e_1(x))\varphi'(x)=\pi_1(\varphi)e_1(x)$$
$$2e_0(x)(e_1(x)-e_0(x))\varphi'(x)=\pi_0(\varphi)e_1(x)$$

where $\pi_i(\varphi)=\alpha_{2i}\varphi^2+\alpha_{1i}\varphi+\alpha_{0i}$, $i=0,1,2$, are three quadratic polynomials in φ. As a result, we find

$$(\varphi'(x))^2=\tfrac{1}{4}D(\varphi)\tag{4.5}$$

where

$$D(\varphi)=\pi_1^2(\varphi)-4\pi_2(\varphi)\pi_0(\varphi)=\sum_{i=0}^{4}d_i\varphi^i$$

is a generic polynomial in $\varphi(x)$ of degree 4, that is the coefficients d_i are not constrained. The general solution of the differential equation (4.5) can be written in the form

$$\varphi(x)=\gamma\frac{\theta_1(\beta x+u_1)\theta_1(\beta x+u_2)}{\theta_1(\beta x+v_1)\theta_1(\beta x+v_2)}\tag{4.6}$$

where $\theta_1(u)$ is the Jacobi theta function

$$\theta_1(u)=2\sum_{n=0}^{\infty}(-1)^n\,\mathrm{e}^{\pi\,\mathrm{i}\tau(n+1/2)^2}\sin\pi(2n+1)u\tag{4.7}$$

and the only constraints upon the parameters γ, β, $u_{1,2}$, $v_{1,2}$, τ are

$$u_1+u_2=v_1+v_2\qquad \mathrm{Im}(\tau)>0.$$

The first of these conditions guarantees that the meromorphic function $\varphi(x)$ is doubly periodic (i.e., it is an elliptic function). There are in total six free parameters in (4.6). Equation (4.5) contains five parameters d_i, and the sixth parameter, the integration constant x_0, enters solutions via the shift $x\to x+x_0$. By a linear fractional transformation $y=(a\varphi+b)/(c\varphi+d)$, equation (4.5) can be reduced to the form $(y')^2=4y^3-g_2y-g_3$, which is solved in terms of the Weierstrass function $\mathcal{P}(x)$. Therefore $\varphi(x)$ is given by a linear fractional transformation of $\mathcal{P}(x)$ or by the general elliptic function of second order (4.6). In [24, 25], function (4.6) described the zeros of a family of discrete self-dual biorthogonal rational functions.

For $e_1(x)$ and $e_0(x)$, we derive expressions

$$e_1(x) = -\frac{2\varphi'}{\pi_2(\varphi)} \qquad e_0(x) = -\frac{\pi_1(\varphi) + 2\varphi'}{2\pi_2(\varphi)}. \tag{4.8}$$

As a simple explicit example, we consider the case $\pi_1 = 0$ or $\alpha_{i1} = 0, i = 0, 1, 2$, and $\pi_2 = \varphi^2 - 1, \pi_0 = 1 - k^2\varphi^2, 0 < k < 1$. From (4.5), we then find

$$(\varphi'(x))^2 = (1 - \varphi^2)(1 - k^2\varphi^2).$$

The solution of this equation is given by the Jacobi elliptic sine function with modulus $k, \varphi(x) = \mathrm{sn}(x - x_0)$, where x_0 is an arbitrary constant. Other functions of interest are

$$e_0(x) = \frac{1}{2}e_1(x) = \frac{\mathrm{dn}(x - x_0)}{\mathrm{cn}(x - x_0)}.$$

Making the shift $x \to x + x_0$, we can write

$$X = \mathrm{sn}(x) \qquad Y = \frac{\mathrm{dn}(x)}{\mathrm{cn}(x)}\left(1 + \frac{2}{\mathrm{e}^{2y} - 1}\right) = \frac{\mathrm{dn}(x)}{\mathrm{cn}(x)} \coth y.$$

The dual representation $X = \tilde{e}_0(\xi) + \tilde{e}_1(x)/(\mathrm{e}^{2\eta} - 1)$, $Y = \psi(\xi)$ corresponds to the transposed matrix $\tilde{\alpha}_{ik} = \alpha_{ki}$. In the taken special case, we have

$$\psi(\xi) = \frac{1}{\mathrm{sn}(\xi)} \qquad \tilde{e}_0(\xi) = \frac{1}{2}\tilde{e}_1(\xi) = \frac{\mathrm{cn}(\xi)}{\mathrm{dn}(\xi)}$$

and, hence,

$$X = \frac{\mathrm{cn}(\xi)}{\mathrm{dn}(\xi)} \coth \eta \qquad Y = \frac{1}{\mathrm{sn}(\xi)}.$$

We see that the shapes of variables X and Y in the initial and dual pictures almost coincide.

Consider now the degenerate case, when $D(\varphi) \to 0$. In order to have some y-dependence in Y we scale the canonical variables $x \to x/\kappa$, $y \to \kappa y$ and take the limit $\kappa \to 0$. Since, $\mathrm{e}^{2\kappa y} - 1 \to 2\kappa y$ we assume that $\kappa^{-1}e_1(x/\kappa)$ is finite in this limit. As a result, we come to the ansatz

$$X = \varphi(x) \qquad Y = e_0(x) + \frac{e_1(x)}{y + g(x)}$$

that is Y becomes a rational function of y. Using the shift $y \to y - g(x)$ and a change of variable x, Y can be reduced to a simpler form $Y = e_0(x) + y^{-1}$.

In this case, $Z = -\varphi'(x)/y^2 = -\varphi'(x)(Y - e_0(x))^2$, that is the quadratic polynomial $Z = \pi_2(X)Y^2 + \pi_1(X)Y + \pi_0(X)$ has a double zero as a function of Y. This corresponds precisely to the condition $D(X) = \pi_1^2(X) - 4\pi_2(X)\pi_0(X) = 0$. Obviously, this can happen only if both $\pi_0(X)$ and $\pi_2(X)$ are complete squares or $\pi_2(X) = \gamma\pi_0(X)$ for some constant γ. We also have

$$\varphi'(x) = -\pi_2(\varphi) \qquad e_0(x) = -\frac{\pi_1(\varphi)}{2\pi_2(\varphi)} \tag{4.9}$$

that is the 'grid' $\varphi(x)$ and the 'potential' $e_0(x)$ are expressed in terms of elementary functions.

Other types of simplified situations are described by the choices

$$X = \varphi(x) \qquad Y = \frac{\mathrm{e}^y + g(x)}{1 + b(x)\,\mathrm{e}^{-y}} \tag{4.10}$$

corresponding to Laurent biorthogonal polynomials, and its generalization

$$X = \varphi(x) \qquad Y = \frac{\mathrm{e}^y + g(x) + f(x)\,\mathrm{e}^{-y}}{1 + b(x)\,\mathrm{e}^{-y}}. \tag{4.11}$$

We skip these intermediate situations and pass directly to the general case.

5. The general case

Suppose that none of the coefficients $A_i^{(r)}$, $B_i^{(r)}$ in (3.11) vanishes identically. In this case the variables X and Y can be represented in the form

$$X = \varphi(x) \qquad Y = e_0(x) + \frac{e_1(x)}{\mathrm{e}^y - g_1(x)} + \frac{e_2(x)}{\mathrm{e}^y - g_2(x)} \tag{5.1}$$

and in the dual picture

$$Y = \psi(\xi) \qquad Y = t_0(\xi) + \frac{t_1(\xi)}{\mathrm{e}^\eta - r_1(\xi)} + \frac{t_2(\xi)}{\mathrm{e}^\eta - r_2(x)}. \tag{5.2}$$

By means of the canonical transformations $y \to y + f(x)$ we can normalize $g_1(x) = 1$ or $g_2(x) = 1$, but we keep $g_{1,2}$ arbitrary for symmetry reasons. The limit $g_1 \to g_2$ is degenerate since in this case $Y(x, y)$ can have a double pole in e^y.

Theorem 2. *Suppose that independent variables X and Y admit representations (5.1) and (5.2) via two canonical transformations and denote $Z \equiv \{X, Y\}$. Then X, Y, Z necessarily satisfy quadratic equations determining particular elliptic curves*

$$Z^2 + F_1(X, Y)Z + F_2(X, Y) = 0 \qquad Z^2 + G_1(X, Y)Z + G_2(X, Y) = 0 \tag{5.3}$$

where F_1 and G_1 are polynomials of the second degree in Y and X, respectively, and

$$F_2(X, Y) = \tfrac{1}{4}F_1(F_1 - q^2) \qquad G_2(X, Y) = \tfrac{1}{4}G_1(G_1 - r^2) \tag{5.4}$$

where $q = q(X, Y)$ and $r = r(X, Y)$ are polynomials in Y and X of the first degree, respectively.

Proof. First we compute $Z = \{X, Y\}$ in the representation (5.1)

$$Z = -\varphi'(x)\,\mathrm{e}^y \left(\frac{e_1(x)}{(\mathrm{e}^y - g_1(x))^2} + \frac{e_2(x)}{(\mathrm{e}^y - g_2(x))^2} \right). \tag{5.5}$$

Excluding e^y from Z and Y using the resultant technique, we obtain the first elliptic curve equation in (5.3) with

$$\begin{aligned} F_1(X, Y) = {} & \frac{\varphi'(x)(e_1 g_2 + e_2 g_1)}{e_1 e_2 (g_2 - g_1)^2}((g_2 - g_1)^2 (Y - e_0)^2 \\ & + 2(g_2 - g_1)(e_2 - e_1)(Y - e_0) + (e_1 + e_2)^2) \end{aligned} \tag{5.6}$$

and

$$F_2(X, Y) = \frac{\varphi'(x)(Y - e_0)(g_1 g_2 (Y - e_0) + e_1 g_2 + e_2 g_1)}{e_1 g_2 + e_2 g_1} F_1(X, Y). \tag{5.7}$$

As usual, it is assumed that the function $\varphi(x)$ is invertible, and we can substitute $x = x(X)$ in the right-hand sides of (5.6) and (5.7).

Evidently, $F_1(X, Y)$ is a quadratic polynomial in Y, whereas $F_2(X, Y)$ is a special quartic polynomial in Y equal to the product of F_1 and of another quadratic polynomial in Y. Calculating the discriminant $D(X, Y)$ of the quadratic equation in Z (5.3), we obtain

$$D(X, Y) \equiv F_1^2 - 4F_2 = q^2(X, Y) F_1(X, Y) \tag{5.8}$$

where $q(X, Y)$ is a linear function of Y of the following form:

$$q(X, Y) = \left(\frac{\varphi'(x)}{e_1 e_2 (e_1 g_2 + e_2 g_1)} \right)^{1/2} \left((e_2 g_1 - e_1 g_2)(Y - e_0) + \frac{e_1 + e_2}{g_2 - g_1}(e_2 g_1 + e_1 g_2) \right). \tag{5.9}$$

Thus $D(X, Y)$ has a double zero in Y. Using (5.8), we can rewrite F_2 in the required form (5.4) with $F_1(X, Y)$ given by (5.6) and $q(X, Y)$ fixed in (5.9).

Now we can repeat the above considerations in the dual representation (5.2). From the permutational symmetry it follows that $G_1(X, Y)$ is now a quadratic polynomial in X and $r(X, Y)$ is a linear function in X. The theorem is proved. □

In order to find explicit forms of functions F_1, G_1, q, and r it is necessary to solve the resultant equation, obtained after exclusion of Z from equations (5.3),

$$(G_2 - F_2)^2 = (F_1 - G_1)(G_1 F_2 - G_2 F_1).$$

We do not know all its solutions and limit consideration to the obvious case $F_1 = G_1$ and $F_2 = G_2$ (i.e., $q = r$). This leads to a quite general situation when $F_1(X, Y)$ and $q(X, Y)$ are polynomial in both X and Y

$$F_1(X, Y) = \sum_{i,k=0}^{2} \alpha_{ik} X^i Y^k \qquad q(X, Y) = \sum_{i,k=0}^{1} \beta_{ik} X^i Y^k \tag{5.10}$$

for some 13 constants α_{ik} and β_{ik}. It is not clear whether all the latter constants are independent or if they must satisfy some additional constraints. Let us prove that in general there are no constraints upon them.

Theorem 3. *Suppose that two independent variables X, Y satisfy the condition (5.3), where $Z = \{X, Y\}$ and $F_1(X, Y)$, $F_2(X, Y)$ are given by (5.10), (5.4) for arbitrary constants α_{ik}, β_{ik}. Then there exist two canonical transformations $(q, p) \to (x, y)$ and $(q, p) \to (\xi, \eta)$ yielding representation (5.1) and its dual (5.2), respectively.*

Proof. We choose a pair of new canonical variables (x, y) in such a way that X coincides with x: $X = x$, $Y = f(x, y)$, so that $Z = \{X, Y\} = f_y(x, y)$. From relation (5.3), we obtain a nonlinear differential equation for $f(x, y)$

$$f_y^2(x, y) + F_1 f_y(x, y) + \tfrac{1}{4} F_1 (F_1 - q^2) = 0 \tag{5.11}$$

where

$$\begin{aligned} F_1(x, f) &= \pi_2(x) f^2 + \pi_1(x) f + \pi_0(x) \\ \pi_i(x) &= \alpha_{2i} x^2 + \alpha_{1i} x + \alpha_{0i} \qquad i = 0, 1, 2 \end{aligned} \tag{5.12}$$

and

$$q(x, f) = \tau_1(x) f + \tau_0(x) \qquad \tau_i(x) = \beta_{1i} x + \beta_{0i} \qquad i = 0, 1. \tag{5.13}$$

We denote $D \equiv \pi_1^2 - 4\pi_0\pi_2$ and substitute into equation (5.11) the ansatz

$$f(x, y) = \frac{\sqrt{D(x)}}{4\pi_2(x)} (u(x, y) + u^{-1}(x, y)) - \frac{\pi_1(x)}{2\pi_2(x)} \tag{5.14}$$

where $u(x, y)$ is an unknown function. After dropping the common factor $D(1-u^{-2})^2/16\pi_2^2$, we obtain

$$u_y^2 + \frac{1}{4}\sqrt{D}(u^2 - 1) u_y + \frac{D}{64\pi_2}(u^2 - 1)^2 - \frac{1}{4}\pi_2 u^2 q^2 = 0.$$

Resolving this quadratic equation with respect to u_y, we arrive at

$$u_y(x, y) = \tfrac{1}{8}\sqrt{D}(1 - u^2) \pm \tfrac{1}{2}\sqrt{\pi_2} u q.$$

Since the function $uq(x, f)$ is a quadratic polynomial in u, we can rewrite this equality as

$$u_y(x, y) = \kappa_2(x) u^2(x, y) + \kappa_1(x) u(x, y) + \kappa_0(x) \tag{5.15}$$

for some functions $\kappa_i(x)$, $i = 0, 1, 2$.

The general solution of linear ordinary differential equation (5.15) is

$$u(x, y) = \epsilon_1(x)\tanh\left(\tfrac{1}{2}\omega(x)(y - y_0(x))\right) + \epsilon_0(x)$$

where $y_0(x)$ is an arbitrary function of x and

$$\epsilon_0 = -\frac{\kappa_1}{2\kappa_2} \qquad \omega = -\sqrt{\kappa_1^2 - 4\kappa_2\kappa_0} \qquad \epsilon_1 = -\frac{\omega}{2\kappa_2}.$$

Returning to the original variable $Y = f(x, y)$, we can represent the derived solution (5.14) in the form

$$f(x, y) = \frac{V_1(x)\,\mathrm{e}^{\omega(x)(y-y_0)} + V_2(x)\,\mathrm{e}^{-\omega(x)(y-y_0)} + V_0(x)}{W_1(x)\,\mathrm{e}^{\omega(x)(y-y_0)} + W_2(x)\,\mathrm{e}^{-\omega(x)(y-y_0)} + W_0(x)}$$

for some functions $V_i(x)$, $W_i(x)$, $i = 0, 1, 2$. Performing a simple canonical transformation $\omega(x)(y - y_0) \to y$ and $x \to \varphi(x)$, such that $\omega(x)\varphi'(x) = 1$, we come to the needed representation (5.1). The theorem is proved. □

6. Derivation of the potentials

Now it is necessary to find explicit expressions for the grid function $\varphi(x)$ and potentials $e_{0,1,2}(x)$, $g_{1,2}(x)$. Relations (5.6) and (5.7) are central for this purpose. Let us return to the first equation in (5.3), where X, Y are fixed in (5.1) and polynomials F_1, F_2 have the form

$$F_1(X, Y) = \sum_{i,k=0}^{2} \alpha_{ik} X^i Y^k = \pi_2(X)Y^2 + \pi_1(X)Y + \pi_0(X)$$

$$F_2(X, Y) = \frac{1}{4}F_1(F_1 - q^2) \qquad q = \sum_{i,k=0}^{1} \beta_{ik} X^i Y^k = \tau_1(X)Y + \tau_0(X).$$

Introduce special notation for the ratio F_2/F_1 of these expressions

$$\tfrac{1}{4}(F_1 - q^2) \equiv \rho_2(X)Y^2 + \rho_1(X)Y + \rho_0(X) \tag{6.1}$$

where

$$4\rho_2 = \pi_2 - \tau_1^2 \qquad 4\rho_1 = \pi_1 - 2\tau_1\tau_0 \qquad 4\rho_0 = \pi_0 - \tau_0^2. \tag{6.2}$$

Equating F_1 as given above with (5.6), we obtain

$$\varphi' = \pi_2 \frac{e_1 e_2}{e_1 g_2 + e_2 g_1} \tag{6.3}$$

$$\frac{e_2 - e_1}{g_2 - g_1} = e_0 + \frac{\pi_1}{2\pi_2} \tag{6.4}$$

$$\frac{(e_1 + e_2)^2}{(g_2 - g_1)^2} = \frac{\pi_0 + e_0\pi_1 + e_0^2\pi_2}{\pi_2}. \tag{6.5}$$

Similarly, equating (6.1) with the ratio F_2/F_1 appearing from (5.7), we derive

$$\varphi' = \rho_2\left(\frac{e_1}{g_1} + \frac{e_2}{g_2}\right) \tag{6.6}$$

$$\frac{e_1}{g_1} + \frac{e_2}{g_2} = 2e_0 + \frac{\rho_1}{\rho_2} \tag{6.7}$$

$$e_0\left(e_0 - \frac{e_1}{g_1} - \frac{e_2}{g_2}\right) = \frac{\rho_0}{\rho_2}. \tag{6.8}$$

The latter set of equations yields

$$(\varphi'(x))^2 = \rho_1^2(\varphi) - 4\rho_0(\varphi)\rho_2(\varphi) \tag{6.9}$$

$$e_0(x) = \frac{\varphi'(x) - \rho_1(\varphi)}{2\rho_2(\varphi)} \qquad \frac{e_1(x)}{g_1(x)} + \frac{e_2(x)}{g_2(x)} = \frac{\varphi'(x)}{\rho_2(\varphi)}. \tag{6.10}$$

On the right-hand side of (6.9) we have a fourth degree polynomial of φ. Therefore, the following general statement is valid.

Proposition 4. *The 'grid' function $\varphi(x)$ is an elliptic function satisfying differential equation (6.9). It is equal to the ratio of four Jacobi theta functions as described in (4.6).*

The potential $e_0(x)$ is given by a simple combination of $\varphi'(x)$ and $\varphi(x)$ (i.e., it is also an elliptic function). Combining (6.3) with (6.6), we obtain

$$\frac{e_1 e_2}{g_1 g_2} = \frac{(\varphi')^2}{\pi_2\rho_2}.$$

This relation and the second equation in (6.10) yield

$$e_{1,2} = g_{1,2}\frac{\varphi'}{2\rho_2}\left(1 \pm \frac{\tau_1}{\sqrt{\pi_2}}\right). \tag{6.11}$$

Substitution of the derived expressions for $e_{0,1,2}$ into equation (6.4) results in

$$\frac{g_2 + g_1}{g_2 - g_1} = \frac{\pi_1\tau_1 - 2\pi_2\tau_0}{4\varphi'\sqrt{\pi_2}} \tag{6.12}$$

from where one determines the ratio g_1/g_2. As seen, the potentials $e_{1,2}$ and $g_{1,2}$ involve the square root of the polynomial $\pi_2(\varphi)$. Therefore, unless π_2 is a complete square, these potentials are not given by elliptic functions (they are double periodic but not meromorphic functions of x). However, in the non-factorized expression

$$Y = \frac{e_0\,\mathrm{e}^{y} + (e_0 g_1 g_2 - e_1 g_2 - e_2 g_1)\,\mathrm{e}^{-y} + e_1 + e_2 - (g_1 + g_2)e_0}{\mathrm{e}^{y} + g_1 g_2\,\mathrm{e}^{-y} - g_1 - g_2} \tag{6.13}$$

we have only particular combinations of e_i and g_i. Using the gauge freedom $y \to y + f(x)$, we can fix $g_1 = g_2 + \sqrt{\pi_2}\chi(\varphi)$, where $\chi(\varphi)$ is a polynomial or rational function of $\varphi(x)$. As a result, all potentials in (6.13) become elliptic functions.

Since all entering quantities were determined already, equation (6.5) must be an identity. This is easily verified with the help of (6.9) and (6.10). There is another way of fixing potentials. We could start by equating $q(X, Y)$, as fixed in (5.9), with $\tau_1 Y + \tau_0$. This brings in only five equations, making it evident that one of the equations (6.3)–(6.5) must be an identity. However, technically speaking, this way of derivation appears to be essentially more complicated than the present one.

7. A generalization of Sklyanin's Poisson algebra

We have shown that when $F_1 = G_1$ and $F_2 = G_2$ conditions (5.3) are necessary and sufficient for dynamical variables X and Y to satisfy a classical analogue of the generalized eigenvalue problem for two tridiagonal matrices. The corresponding Poisson algebra takes the form

$$\{X, Y\} = Z \qquad \{Z, X\} = -Z\frac{\partial Z}{\partial Y} \qquad \{Y, Z\} = -Z\frac{\partial Z}{\partial X}. \tag{7.1}$$

However, the variable Z considered as a function of X, Y is given by a root of a quadratic equation. The algebra obtained after the substitution $Z = (-F_1 \pm q\sqrt{F_1})/2$ into (7.1) has a much less attractive form with respect to the Askey–Wilson case (2.6). Therefore, we need to find some 'aesthetic simplification' procedure. For instance, we may try to express X, Y, Z in terms of some other dynamical variables $U_i, i = 1, 2, \ldots,$ so that the algebra (7.1) is reproduced by relatively simple Poisson algebraic relations between generators U_i.

This idea can be explained as follows. From the very beginning we have demanded that in the picture (5.1) the variable Y is presented as the ratio $Y = U_3/U_4$ where both U_3 and U_4 are 'tridiagonal' in y. Similarly, in the dual picture (5.2) one should be able to represent $X = U_1/U_2$, where U_1, U_2 have the same tridiagonality property with respect to η. It is therefore natural to set

$$X = \frac{U_1(q,p)}{U_2(q,p)} \qquad Y = \frac{U_3(q,p)}{U_4(q,p)}$$

and seek four dynamical variables U_1, U_2, U_3, U_4 such that for some canonical transformation $(q,p) \to (x,y)$ all U_i are reduced to the tridiagonal form and, additionally, $U_1 = \varphi(x)U_2$ for some function $\varphi(x)$. Similarly, there should exist a dual canonical transformation $(q,p) \to (\xi,\eta)$ which reduces again all U_i to tridiagonal form and, additionally, guarantees that $U_3 = \psi(\xi)U_4$ for some function $\psi(\xi)$. All these requirements bring nothing new to the picture we have considered so far. The crucial additional requirements look as follows:

(i) pairwise Poisson brackets of $\{U_i, U_j\}$ should be quadratic polynomials in U_i;

(ii) the linear transformations $\tilde{U}_1 = m_{11}U_1 + m_{12}U_2$, $\tilde{U}_2 = m_{21}U_1 + m_{22}U_2$ and $\tilde{U}_3 = \ell_{11}U_3 + \ell_{12}U_4$, $\tilde{U}_4 = \ell_{21}U_3 + \ell_{22}U_4$ with two arbitrary nonsingular matrices ℓ_{ij}, m_{ij} do not change the form of the Poisson algebra for U_i (i.e., it should be covariant with respect to such linear transformations).

Condition (ii) can be explained as follows. The permutation $U_1 \leftrightarrow U_2$ is equivalent to the transformation $\tilde{X} = 1/X$, $\tilde{Y} = Y$. This leads to $\tilde{Z} = -Z/X^2$ and it can be verified that if X, Y, Z satisfy condition (5.3) (with $F_2 = F_1(F_1 - q^2)/4$) then the new variables $\tilde{X}, \tilde{Y}, \tilde{Z}$ satisfy a similar equation with changed functions $\tilde{F}_1, \tilde{F}_2$ satisfying the constraint $\tilde{F}_2 = \tilde{F}_1(\tilde{F}_1 - \tilde{q}^2)/4$. Thus, if a pair (X, Y) satisfies the GLD property then the same is true for the pairs $(1/X, Y), (X, 1/Y), (1/X, 1/Y)$. As a result, general linear fractional transformations

$$\tilde{X} = \frac{aX + b}{cX + d} \qquad \tilde{Y} = \frac{AY + B}{CY + D}$$

do not change the form of the elliptic curve equations (5.3).

Some generalizations of the Sklyanin algebra were discussed in [14]. They are generated by two polynomials $Q_1(U)$ and $Q_2(U)$ depending on four dynamical variables $U_i, i = 1, \ldots, 4$, whose Poisson brackets are defined in the following nice way:

$$\{U_i, U_j\} = (-1)^{i+j} \det\left(\frac{\partial Q_k}{\partial U_l}\right) \qquad l \neq i, j, \quad i > j. \tag{7.2}$$

The functions Q_1, Q_2 serve as Casimir elements of this algebraic relations, that is $\{U_i, Q_k\} = 0, k = 1, 2$.

In order to get a quadratic Poisson algebra it is necessary to fix $Q_1(U)$ and $Q_2(U)$ as quadratic polynomials. For example, the standard Sklyanin's Poisson algebra is obtained from

$$Q_1(U) = U_1^2 + U_2^2 + U_3^2 \qquad Q_2(U) = U_4^2 + J_1U_1^2 + J_2U_2^2 + J_3U_3^2.$$

This construction can be used to model our classical Poisson algebraic relations (5.3). We consider only the case of two-diagonal representation (4.4). Calculating the Poisson bracket of $X = U_1/U_2$, $Y = U_3/U_4$, we obtain

$$Z = \{X, Y\} = \frac{\{U_1, U_3\}}{U_2U_4} + \frac{U_1U_3\{U_2, U_4\}}{U_2^2U_4^2} - \frac{U_3\{U_1, U_4\}}{U_2U_4^2} - \frac{U_1\{U_2, U_3\}}{U_2^2U_4}. \tag{7.3}$$

It is seen that in order to obtain (4.4) it is necessary to demand that each Poisson bracket $\{U_i, U_j\}$ in (7.3) contains U_iU_j for the same i and j, the adjacent pair U_kU_l (where k or l are not equal to i or j) and two adjacent pairs U_iU_k and U_lU_j. These conditions can be satisfied if we choose Casimir elements in the form

$$\begin{aligned} Q_1(U) &= \sum_{i=1}^{4} a_i U_i^2 + \xi_1 U_1 U_2 + \eta_1 U_3 U_4 \\ Q_2(U) &= \sum_{i=1}^{4} b_i U_i^2 + \xi_2 U_1 U_2 + \eta_2 U_3 U_4 \end{aligned} \tag{7.4}$$

with arbitrary parameters a_i, b_i, ξ_m, η_m. Direct calculations show that

$$\begin{aligned} \tfrac{1}{4} Z = (a_3 b_1 - a_1 b_3) X^2 Y^2 &+ (a_3 \xi_2 - b_3 \xi_1) X Y^2 + (b_1 \eta_1 - a_1 \eta_2) X^2 Y \\ &+ (\eta_1 \xi_2 - \eta_2 \xi_1) X Y + (a_3 b_2 - b_3 a_2) Y^2 + (a_4 b_1 - a_1 b_4) X^2 \\ &+ (b_2 \eta_1 - a_2 \eta_2) Y + (a_4 \xi_2 - b_4 \xi_1) X + a_4 b_2 - a_2 b_4. \end{aligned}$$

We note that property (ii) is fulfilled (e.g., the permutation $U_1 \leftrightarrow U_2$ leads only to the permutation of parameters a_1, a_2 and b_1, b_2 in Q_1 and Q_2).

Theorem 5. *Any symmetric generalized CLP with $X = U_1/U_2$ and $Y = U_3/U_4$ can be realized in terms of the quadratic Poisson algebra (7.2) with two Casimir elements given by (7.4).*

In a different context, the relevance of the Sklyanin algebra for discrete elliptic biorthogonal functions has been noted recently by Rains [16] and Rosengren [17].

The general non-symmetric systems obeying the generalized classical Leonard duality are essentially more complicated and require a separate consideration.

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